When is a one-point space computably metrizable?

Arno Pauly

Swansea University

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The question

Definition

A represented space **X** is computably metrizable if there is a computable embedding $\iota : \mathbf{X} \hookrightarrow [0, 1]^\omega$.

Observation

There are (up to isomorphism) $2^{\mathfrak{c}}$-many one-point represented spaces, but only $\mathfrak{c}$ of them are computably metrizable.

Observation

The effectively countably-based one-point represented spaces correspond to the enumeration degrees. A space is computably metrizable iff the corresponding degree is a continuous degree.

An absurd characterization

Theorem (Joe Miller & Co)

A represented space $\{x\}$ *is computably metrizable iff for every* $p \in \{0,1\}^{\mathbb{N}}$ *the space* $\{x\} \times \{p\}$ *embeds into* $\{x\} \sqcup \{0,1\}^{\mathbb{N}}$.

Some topological properties trivialize

Proposition

Every one-point space is T_1, computably Hausdorff and computably discrete.

Proposition

The following are equivalent for a computably admissible one-point space $\{x\}$:

1. $\{x\}$ *is computably compact.*
2. $\{x\}$ *is computably overt.*
3. $\{x\} \cong \mathbf{1}$

What about computably normal?

Observation

The space $\{A\} \subseteq \mathcal{O}(\mathbb{N})$ *is computably normal iff there is a computable procedure that upon receiving enumerations of* B_0, B_1 *outputs enumerations of* C_1, C_2 *such that:*

1. $C_1 \cap A = \emptyset$ *or* $C_2 \cap A = \emptyset$
2. *If* $B_i \cap A \neq \emptyset$ *and* $B_{1-i} \cap A = \emptyset$, *then* $C_{i-1} \cap A \neq \emptyset$

Does this get us anywhere?